Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 313 (2006) 654–677
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

An efficient discretization scheme for solving ill-posed problems

M.P. Rajan

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati 781 039, Assam, India

Received 13 February 2004
Available online 12 July 2005
Submitted by M.D. Gunzburger

Abstract

In this paper, we consider a finite-dimensional approximation scheme combined with Tikhonov regularization for solving ill-posed problems. Error estimates are obtained by an a priori parameter choice strategy and the results show that the amount of discrete information required for solving the problem is far less than the traditional finite-dimensional approach.

Keywords: Ill-posed problems; Tikhonov regularization

1. Introduction

It is well known (cf. [4]) that Tikhonov regularization is one of the useful tools for solving an ill-posed problem of the form

$$Ax = y, \tag{1.1}$$

where A is a bounded linear operator between suitable function spaces. In Tikhonov regularization, what one solves is a family of well-posed equations

$$(A^*A + \alpha I)x_\alpha = A^*y, \quad \alpha > 0, \tag{1.2}$$

E-mail address: rajanmp@iitg.ernet.in.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.06.009

and if the data y is not available exactly, but an approximation of it, namely $\tilde{y}$, is available, then one solves

$$(A^*A + \alpha I)\tilde{x}_\alpha = A^*\tilde{y}, \quad \alpha > 0, \tag{1.3}$$

instead of (1.2). The necessity of regularization procedures for the problem (1.1) arise from the fact that even the problem of finding the generalized solution $\hat{x} = A^\dagger y$, where $A^\dagger$ is the Moore–Penrose generalized inverse of A, can be ill-posed if $R(A)$ is not closed. One of the well-known examples for (1.1) is the Fredholm integral equations of the first kind. In the sequel, we assume that $A : X \to X$ is a linear compact operator acting on a Hilbert space X. If one employs Tikhonov regularization as a tool for solving (1.1), it can be seen using spectral results that (cf. [4])

$$\|\hat{x} - x_\alpha\| \to 0 \text{ as } \alpha \to 0 \quad \text{and} \quad \|x_\alpha - \tilde{x}_\alpha\| \leqslant \frac{\|y - \tilde{y}\|}{2\sqrt{\alpha}}.$$

Moreover, if $\hat{x} \in R((A^*A)^\nu)$, $0 < \nu \leqslant 1$ and $\|y - \tilde{y}\| \leqslant \delta$, for some $\delta > 0$, then by choosing $\alpha \sim \delta^{2/(2\nu+1)}$ we have

$$\|\hat{x} - \tilde{x}_\alpha\| = O\big(\delta^{2\nu/(2\nu+1)}\big).$$

It is known (cf. [4]) that the above rate is optimal and the best rate possible for Tikhonov regularization is $O(\delta^{2/3})$ which is achieved for $\nu = 1$. In applications, Eq. (1.3) is usually solved using numerical procedures. In such cases, one considers an approximated form of (1.3), namely,

$$(A_n^*A_n + \alpha I)\tilde{x}_{\alpha,n} = A_n^*\tilde{y}, \quad \alpha > 0, \tag{1.4}$$

with $\|A_n - A\| \to 0$ as $n \to \infty$. For example, in projection method, one looks for an element $\tilde{x}_{\alpha,n}$ in a finite-dimensional subspace X_n of X such that

$$\langle A^*A\tilde{x}_{\alpha,n} + \alpha\tilde{x}_{\alpha,n}, u\rangle = \langle A^*\tilde{y}, u\rangle \quad \forall u \in X_n. \tag{1.5}$$

This is equivalent to solving the equation

$$(P_nA^*AP_n + \alpha I)\tilde{x}_{\alpha,n} = P_nA^*\tilde{y}, \tag{1.6}$$

where $P_n : X \to X$ is the orthogonal projection onto X_n. If X is separable and $\{e_1, e_2, \ldots\}$ is an orthonormal basis of X, then P_n may be given by

$$P_nx = \sum_{j=1}^{n} \langle x, e_j\rangle e_j, \quad x \in X.$$

Since each P_n is of finite rank, Eq. (1.6) can be reduced to a matrix equation which can be solved numerically. Recent works of P. Maass et al. [5] shows that proper choice of the approximation schemes can result in better advantage in terms of computational complexity than the traditional approaches. One can construct a finite-dimensional operator in such a way that it not only preserves the optimality properties but also has computational advantage with respect to the amount of discrete information required for computing the solution. In the following, for $r > 0$, let X_r be a dense subspace of the Hilbert space X with the norm

$$\|f\|_r := \|f\| + \|L_rf\|, \quad f \in X_r,$$

where $L_r : X \subset X_r \to X$ is a closed linear operator. If $A : X \to X$, $B : X_r \to X$ and $C : X \to X_r$ are bounded operators, we shall denote their norms by

$$\|A\|, \qquad \|B\|_{r,0}, \qquad \|C\|_{0,r},$$

respectively.

We assume that the operator $A : X \to X$ has certain *smoothness properties*. We consider two sets of properties namely,

$$S_1 = \{\|A\|_{0,r} \leqslant \gamma_1,\ \|A^*\|_{0,r} \leqslant \gamma_2\} \tag{1.7}$$

and

$$S_2 = \{\|A\|_{0,r} \leqslant \gamma_1,\ \|A^*\|_{0,r} \leqslant \gamma_2,\ \|(L_r A)^*\|_{0,r} \leqslant \gamma_3\}, \tag{1.8}$$

where γ_1, γ_2, γ_3 are positive real numbers, and carry out our analysis in two cases with respect to these smoothness properties.

An approximation scheme under our consideration is

$$(A_m^* A_m + \alpha I)\tilde{x}_{\alpha,m} = A_m^* \tilde{y}, \quad \alpha > 0,$$

with

$$A_m = P_0 A P_m + \sum_{k=1}^{m} (P_k - P_{k-1}) A P_{m-k}, \tag{1.9}$$

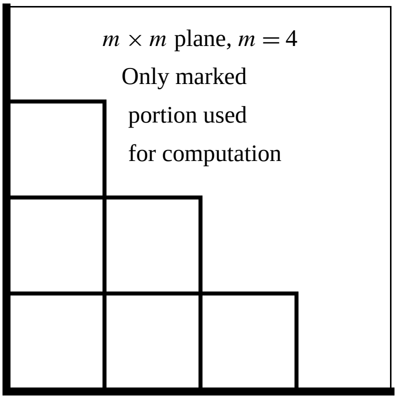
where (P_j) defined by $P_j x = \sum_{j=1}^{\dim V_j} \langle x, e_j \rangle e_j$, $x \in X$, is a sequence of orthogonal projections on the finite-dimensional subspaces $V_0 \subset V_1 \subset \cdots \subset V_j \subset \cdots \subset X$ with $\dim V_j \sim 2^{sj}$, $s \geqslant 1$. Further we assume that the sequence of projections has the *approximation property*,

$$\|I - P_j\|_{r,0} \leqslant \kappa_r 2^{-rj} \quad \text{for all } j \in N,\ r > 0, \tag{1.10}$$

where $\kappa_r \geqslant 1$ is independent of j and each (i, j) corresponds to (e_i, Ae_j) and $(e_i, \tilde{y})$ lie in a finite plane

$$\Omega_m = \{0\} \times [0, m] \bigcup_{k=1}^{m} (k-1, k] \times [0, m-k].$$

The left-hand side of (1.9) should be treated as a notation. The advantage of these kind of schemes in comparison with traditional projection schemes is that instead of using all the discrete information on a plane with $m \times m$ points, one only needs a portion of these discrete information for computation of the solution (see the figure below). The basic idea of these type of construction comes from the wavelet analysis (cf. [1]).



Apart from discussing the computational advantage of our scheme over the traditional schemes, the significance of the consideration of two smoothness properties for our analysis is two-fold. First, we want to emphasise the fact that with the specific smoothness property, our scheme can be considered as an approximation of A with the weak condition $\|(A_m - A)A\| \to 0$ instead of the strong condition $\|A_m - A\| \to 0$, as $m \to \infty$. Further, our scheme also satisfies properties like $\|A_m^* A_m - A^* A\| \to 0$, $\|(A_m^* - A^*)A\| \to 0$ as $m \to \infty$. Secondly, we want to point out that the smoothness of the operator can result in better estimates and really exploits the specific structure of the discretization scheme.

The smoothness assumptions and approximation property can be illustrated through an example by taking A as an integral operator,

$$(Ax)(s) = \int_0^1 k(s,t)x(t)\,dt, \quad s \in [0,1],$$

on the Hilbert space $X = L^2[0,1]$. Taking $r = 1$, let X_r be the Sobolev space of functions f, with the derivative $f' \in L^2[0,1]$, $L_r = \frac{d}{dt}$ and P_j be the orthogonal projection onto the span of $\{e_1, e_2, \ldots, e_j\}$, where $\{e_i\}$ is an orthonormal basis of Haar-wavelet functions in X, namely, $e_1(t) = 1$ for all $t \in [0,1]$, and for $m = 2^{k-1} + j$, $k = 1, 2, \ldots,$ $j = 1, 2, \ldots, 2^{k-1}$,

$$e_m(t) = \begin{cases} 2^{\frac{k-1}{2}} & \text{if } t \in \left[\frac{j-1}{2^{k-1}}, \frac{j-1/2}{2^{k-1}}\right), \\ -2^{\frac{k-1}{2}} & \text{if } t \in \left[\frac{j-1/2}{2^{k-1}}, \frac{j}{2^{k-1}}\right), \\ 0 & \text{if } t \notin \left[\frac{j-1}{2^{k-1}}, \frac{j}{2^{k-1}}\right]. \end{cases}$$

In such situations, it can be shown that $\|I - P_j\|_{r,0} \leqslant \kappa_r 2^{-rj}$. Moreover, if the kernel $k(s,t)$ of the integral operator has mixed partial derivatives and

$$\int_0^1 \int_0^1 \frac{\partial^{i+j} k(s,t)}{\partial s^i \partial t^j}\,ds\,dt < \infty, \quad i, j = 0, 1,$$

then one can see that the required smoothness properties of the operator are satisfied.

2. Error estimate

In this section, we carry out the convergence analysis of the proposed scheme and obtain the error estimate under specific smoothness assumptions. In order to carry out our analysis, we require few results which will be derived as follows.

Lemma 2.1. *Suppose A satisfies the smoothness property S_1, then for every $m \in N$,*

$$\big\|(I - P_m)A\big\| = O(2^{-mr}), \tag{2.1}$$

$$\big\|A(I - P_m)\big\| = O(2^{-mr}) \tag{2.2}$$

and, in addition, to the above results, if A satisfies the property S_2, we have

$$\big\|A(I - P_m)\big\|_{0,r} = O(2^{-mr}). \tag{2.3}$$

Proof. Proof follows from the smoothness assumptions and the approximation property. □

Lemma 2.2. *If A satisfies the smoothness property S_2, then, for $i, j \in N$,*

$$\big\|A^*(P_i - I)A(P_j - I)\big\| = O\big(2^{-(2i+j)r}\big).$$

Proof. We observe that

$$\begin{aligned}\big\|A^*(P_i - I)A(P_j - I)\big\| &= \big\|A^*(P_i - I)(P_i - I)A(P_j - I)\big\| \\ &\leqslant \big\|A^*(P_i - I)\big\|\big\|(P_i - I)A(P_j - I)\big\| \\ &\leqslant \big\|(P_i - I)A\big\|\big\|(P_i - I)\big\|_{r,0}\big\|A(P_j - I)\big\|_{0,r}.\end{aligned}$$

Therefore, by using Lemma 2.1, we get

$$\big\|A^*(P_i - I)A(P_j - I)\big\| = O(2^{-ir}2^{-ir}2^{-jr}) = O\big(2^{-(2i+j)r}\big). \qquad \square$$

The following result shows that our scheme does not satisfy $\|A - A_m\| \to 0$ but satisfies $\|(A - A_m)A\| \to 0$ as $m \to \infty$ with specific smoothness assumption.

Lemma 2.3. *Suppose A satisfies the smoothness property S_1 then*

$$\|A - A_m\| = O(1) \quad \text{and} \quad \big\|(A - A_m)A\big\| = O(m2^{-mr}).$$

Proof. We observe that

$$A - A_m = (I - P_m)A + (P_m A - A_m). \tag{2.4}$$

Also,

$$P_m A - A_m = P_0 A(I - P_m) + \sum_{k=1}^{m}(P_k - P_{k-1})A(I - P_{m-k}). \tag{2.5}$$

Hence, by using the relation (1.10) and the result in Lemma 2.1 with property S_1, we have

$$\big\|(I - P_m)A\big\| = O(2^{-mr})$$

and

$$\begin{aligned}
\|P_mA - A_m\| &\leqslant \left\|P_0A(I-P_m)\right\| + \sum_{k=1}^{m}\left\|P_k(P_{k-1}-I)A(I-P_{m-k})\right\| \\
&\leqslant \left\|A(I-P_m)\right\| + \sum_{k=1}^{m}\left\|A(I-P_{m-k})\right\| \leqslant \sum_{k=0}^{m}\left\|A(I-P_{m-k})\right\| \\
&\leqslant c\sum_{k=0}^{m}2^{-(m-k)r}, \quad \text{for some } c>0, \\
&\leqslant c2^{-mr}\sum_{k=0}^{m}2^{kr} = O(1).
\end{aligned}$$

Thus, we have $\|A - A_m\| = O(1)$. Again,

$$(A-A_m)A = (I-P_m)AA + (P_mA - A_m)A. \tag{2.6}$$

Also,

$$(P_mA - A_m)A = P_0A(I-P_m)A + \sum_{k=1}^{m}(P_k - P_{k-1})A(I-P_{m-k})A. \tag{2.7}$$

Hence, by using the relation (1.10) and the result in Lemma 2.1 with property S_1, we have

$$\left\|(I-P_m)AA\right\| \leqslant \left\|(I-P_m)A\right\|\|A\| = O(2^{-mr})$$

and

$$\begin{aligned}
\left\|(P_mA - A_m)A\right\| &\leqslant \left\|P_0A(I-P_m)A\right\| + \sum_{k=1}^{m}\left\|P_k(P_{k-1}-I)A(I-P_{m-k})A\right\| \\
&\leqslant \left\|A(I-P_m)\right\|\left\|(I-P_m)A\right\| + \sum_{k=1}^{m}\left\|(I-P_{k-1})A(I-P_{m-k})A\right\| \\
&\leqslant \left\|A(I-P_m)\right\|^2 + \sum_{k=1}^{m}\left\|(I-P_{k-1})A\right\|\left\|(I-P_{m-k})A\right\| \\
&\leqslant c_1 2^{-2mr} + c_2\sum_{k=1}^{m}2^{-kr}2^{-(m-k)r}, \quad \text{for some } c_1, c_2 > 0, \\
&\leqslant c_1 2^{-2mr} + c_2\sum_{k=0}^{m}2^{-mr} = O(m2^{-mr}).
\end{aligned}$$

Thus, we have $\|(A-A_m)A\| = O(m2^{-mr})$. $\square$

We also want to point out that one can have the strong convergence, $\|A_m - A\| \to 0$ as $m \to \infty$, with the smoothness property S_2. It is an easy exercise to show that $\|A_m - A\| = O(m2^{-mr})$ with the property S_2.

Lemma 2.4. *If A satisfies the smoothness property S_1, then*

$$\left\|(A_m^* - A^*)A\right\| = O(m2^{-mr}) \quad \textit{and} \quad \|A^*A - A_m^*A_m\| = O(m2^{-mr}).$$

If A satisfies the smoothness property S_2, then

$$\left\|(A_m^* - A^*)A\right\| = O(2^{-mr}) \quad \textit{and} \quad \|A^*A - A_m^*A_m\| = O(2^{-mr}).$$

Proof. We will give the estimate for $\|(A_m^* - A^*)A\|$, and the estimates for $\|A^*A - A_m^*A_m\|$ can be derived in a similar manner. The second estimate is given only for reference purpose and not for computing the error estimate. Since

$$A_m = P_0AP_m + \sum_{k=1}^{m}(P_k - P_{k-1})AP_{m-k},$$

we have

$$A_m^* = P_mA^*P_0 + \sum_{k=1}^{m} P_{m-k}A^*(P_k - P_{k-1}).$$

Hence,

$$\begin{aligned}(A_m^* - A^*)A &= P_mA^*P_0A + \sum_{k=1}^{m} P_{m-k}A^*(P_k - P_{k-1})A - A^*A \\ &= \sum_{k=1}^{m}(P_{m-k} - I)A^*(P_k - P_{k-1})A + (P_m - I)A^*P_0A \\ &\quad + A^*(P_m - I)A.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|(A_m^* - A^*)A\right\| &\leqslant \sum_{k=1}^{m}\left\|(P_{m-k} - I)A^*(P_k - P_{k-1})A\right\| + \left\|(P_m - I)A^*P_0A\right\| \\ &\quad + \left\|A^*(P_m - I)A\right\|.\end{aligned}$$

By using Lemma 2.1, we now have

$$\begin{aligned}&\left\|(P_{m-k} - I)A^*(P_k - P_{k-1})A\right\| \\ &\quad = \left\|A^*(P_k - P_{k-1})A(P_{m-k} - I)\right\| \\ &\quad \leqslant \left\|A^*(P_k - I)A(P_{m-k} - I)\right\| + \left\|A^*(I - P_{k-1})A(P_{m-k} - I)\right\| \\ &\quad \leqslant \left\|A^*(P_k - I)\right\|\left\|A(P_{m-k} - I)\right\| + \left\|A^*(I - P_{k-1})\right\|\left\|A(P_{m-k} - I)\right\| \\ &\quad = \left\|(P_k - I)A\right\|\left\|A(P_{m-k} - I)\right\| + \left\|(I - P_{k-1})A\right\|\left\|A(P_{m-k} - I)\right\| \\ &\quad = O(2^{-mr})\end{aligned}$$

and

$$\left\|A^*(P_m - I)A\right\| = O(2^{-2mr}), \tag{2.8}$$

$$\left\|(P_m - I)A^*P_0A\right\| = O(2^{-mr}). \tag{2.9}$$

Therefore,

$$\left\|(A_m^* - A^*)A\right\| \leqslant c_1 \sum_{k=1}^{m} 2^{-mr} + c_2 2^{-mr} + c_3 2^{-2mr} = O(m2^{-mr}).$$

We have

$$\begin{aligned}(A_m^* - A^*)A &= P_m A^* P_0 A + \sum_{k=1}^{m} P_{m-k} A^*(P_k - P_{k-1})A - A^*A \\ &= \sum_{k=1}^{m} (P_{m-k} - I)A^*(P_k - P_{k-1})A + (P_m - I)A^* P_0 A \\ &\quad + A^*(P_m - I)A.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|(A_m^* - A^*)A\right\| &\leqslant \sum_{k=1}^{m} \left\|(P_{m-k} - I)A^*(P_k - P_{k-1})A\right\| + \left\|(P_m - I)A^* P_0 A\right\| \\ &\quad + \left\|A^*(P_m - I)A\right\|.\end{aligned}$$

We observe that

$$\begin{aligned}\left\|(P_{m-k} - I)A^*(P_k - P_{k-1})A\right\| &= \left\|A^*(P_k - P_{k-1})A(P_{m-k} - I)\right\| \\ &\leqslant \left\|A^*(P_k - I)A(P_{m-k} - I)\right\| \\ &\quad + \left\|A^*(I - P_{k-1})A(P_{m-k} - I)\right\|.\end{aligned}$$

By taking $i = k$ and $j = m - k$ in Lemma 2.2 and using S_2, we have

$$\begin{aligned}&\left\|A^*(P_k - I)A(P_{m-k} - I)\right\| = O\big(2^{-(m+k)r}\big), \\ &\left\|A^*(I - P_{k-1})A(P_{m-k} - I)\right\| = O\big(2^{-(m+k)r}\big).\end{aligned}$$

Hence, by using the estimates (2.8) and (2.9), we have

$$\left\|(A_m^* - A^*)A\right\| \leqslant c_1 \sum_{k=1}^{m} 2^{-(m+k)r} + c_2 2^{-mr} + c_3 2^{-2mr} = O(2^{-mr}).$$

Hence the proof is completed. □

Lemma 2.5. *If A satisfies the smoothness property S_1, then*

$$\left\|(A_m - P_m A)A^*\right\| = O(m2^{-mr}).$$

If A satisfies the smoothness property S_2, then

$$\left\|(A_m - P_m A)A^*\right\| = O(2^{-mr}).$$

Proof. From the definition of A_m, we have

$$(A_m - P_m A)A^* = \sum_{k=1}^{m} (P_k - P_{k-1})A(P_{m-k} - I)A^* + P_0 A(P_m - I)A^*. \tag{2.10}$$

Now,

$$(P_k - P_{k-1})A(P_{m-k} - I)A^* = P_k(P_{k-1} - I)A(P_{m-k} - I)A^* \tag{2.11}$$

so that using (1.10) and Lemma 2.1,

$$\begin{aligned}\big\|(P_k - P_{k-1})A(P_{m-k} - I)A^*\big\| &\leqslant \big\|(P_{k-1} - I)A\big\|\,\big\|(P_{m-k} - I)A^*\big\| \\ &= O\big(2^{-kr}2^{-(m-k)r}\big) = O(2^{-mr}).\end{aligned}$$

Also,

$$\big\|P_0 A(P_m - I)A^*\big\| \leqslant \big\|A(P_m - I)\big\|\,\big\|(P_m - I)A^*\big\| = \big\|A(P_m - I)\big\|^2 = O(2^{-2mr}).$$

Therefore,

$$\begin{aligned}\big\|(A_m - P_m A)A^*\big\| &= \big\|P_0 A(P_m - I)A^*\big\| + \sum_{k=1}^{m} \big\|(P_k - P_{k-1})A(P_{m-k} - I)A^*\big\| \\ &= O\left(2^{-2mr} + \sum_{k=1}^{m} 2^{-mr}\right) = O(m2^{-mr}).\end{aligned}$$

If A satisfies the smoothness property S_2 then, using (2.10), (2.11) and Lemma 2.1, we see that

$$\begin{aligned}\big\|(P_k - P_{k-1})A(P_{m-k} - I)A^*\big\| &\leqslant \big\|(P_{k-1} - I)A(P_{m-k} - I)A^*\big\| \\ &\leqslant \big\|(P_{k-1} - I)A(P_{m-k} - I)\big\|\,\big\|(P_{m-k} - I)A^*\big\| \\ &\leqslant \|P_{k-1} - I\|_{r,0}\big\|A(P_{m-k} - I)\big\|_{0,r}\big\|(AP_{m-k} - I)\big\| \\ &= O\big(2^{-kr}2^{-(m-k)r}2^{-(m-k)r}\big) = O\big(2^{-(2m-k)r}\big).\end{aligned}$$

Also,

$$\begin{aligned}\big\|P_0 A(P_m - I)A^*\big\| &\leqslant \big\|A(P_m - I)\big\|\,\big\|(P_m - I)A^*\big\| \leqslant \big\|A(P_m - I)\big\|^2 \\ &= O(2^{-2mr}).\end{aligned}$$

Therefore,

$$\begin{aligned}\big\|(A_m - P_m A)A^*\big\| &= \big\|P_0 A(P_m - I)A^*\big\| + \sum_{k=1}^{m} \big\|(P_k - P_{k-1})A(P_{m-k} - I)A^*\big\| \\ &= O\left(2^{-2mr} + \sum_{k=1}^{m} 2^{-(2m-k)r}\right) \\ &= O\left(2^{-2mr}\sum_{k=0}^{m} 2^{kr}\right) = O(2^{-mr}). \qquad \square\end{aligned}$$

For $\alpha > 0$ and $m \in N$, let

$$R_\alpha = (A^*A + \alpha I)^{-1}A^*, \qquad R_{\alpha,m} = (A_m^*A_m + \alpha I)^{-1}A_m^*,$$

where A and A_m are as given in the last section. For $y \in D(A^\dagger)$, let $\hat{x} = A^\dagger y$ and $\tilde{y} \in X$ be such that $\|y - \tilde{y}\| \leqslant \delta$ for some $\delta > 0$ and also let

$$x_\alpha = R_\alpha y, \qquad \tilde{x}_{\alpha,m} = R_{\alpha,m}\tilde{y}.$$

One can obtain an error estimate using the estimates $\|(A_m^* - A^*)A\|$ and $\|A_m^*A_m - A^*A\|$ as shown in [5] but to get a better estimate we follow a different approach.

We observe that

$$\hat{x} - \tilde{x}_{\alpha,m} = \hat{x} - R_{\alpha,m}\tilde{y} = (\hat{x} - R_\alpha y) + (R_\alpha - R_{\alpha,m})y + R_{\alpha,m}(y - \tilde{y}).$$

Recall from Schock [7] that, if $\hat{x} \in R((A^*A)^\nu)$, $0 < \nu \leqslant 1$, and $\hat{u} \in X$ is such that $\hat{x} = (A^*A)^\nu \hat{u}$, then

$$\|\hat{x} - R_\alpha y\| = \|\hat{x} - x_\alpha\| \leqslant \|\hat{u}\|\alpha^\nu.$$

Using the spectral theory result, $\|(A^*A + \alpha I)^{-1}A^*\| \leqslant \frac{1}{2\sqrt{\alpha}}$ with A_m in place of A, we have

$$\begin{aligned}\big\|R_{\alpha,m}(y - \tilde{y})\big\| &= \big\|(A_m^*A_m + \alpha I)^{-1}A_m^*(y - \tilde{y})\big\| \\ &\leqslant \big\|(A_m^*A_m + \alpha I)^{-1}A_m^*\big\|\big\|(y - \tilde{y})\big\| \leqslant \frac{\delta}{2\sqrt{\alpha}}.\end{aligned}$$

Thus,

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leqslant \|\hat{u}\|\alpha^\nu + \big\|(R_\alpha - R_{\alpha,m})y\big\| + \frac{\delta}{2\sqrt{\alpha}}. \tag{2.12}$$

Now we compute an estimate for $\|(R_\alpha - R_{\alpha,m})y\|$.

Proposition 2.6. *If A satisfies the smoothness property S_1, then*

$$\big\|(R_{\alpha,m} - R_\alpha)y\big\| = O\big(m2^{-mr}\alpha^{\nu-1} + m2^{-mr}\alpha^{\omega-3/2}\big).$$

If A satisfies the smoothness property S_2, then

$$\big\|(R_{\alpha,m} - R_\alpha)y\big\| = O\big(2^{-mr}\alpha^{\nu-1} + 2^{-mr}\alpha^{\omega-3/2}\big).$$

Proof. We observe that

$$\begin{aligned}R_{\alpha,m} - R_\alpha &= (A_m^*A_m + \alpha I)^{-1}A_m^* - (A^*A + \alpha I)^{-1}A^* \\ &= (A_m^*A_m + \alpha I)^{-1}A_m^* - A^*(AA^* + \alpha I)^{-1} \\ &= (A_m^*A_m + \alpha I)^{-1}\big[A_m^*(AA^* + \alpha I) - (A_m^*A_m + \alpha I)A^*\big](AA^* + \alpha I)^{-1} \\ &= (A_m^*A_m + \alpha I)^{-1}\big[\alpha(A_m^* - A^*) + A_m^*(AA^* - A_mA^*)\big](AA^* + \alpha I)^{-1}.\end{aligned}$$

Since $y = A\hat{x}$,

$$\begin{aligned}(R_{\alpha,m}-R_\alpha)y &= \alpha(A_m^*A_m+\alpha I)^{-1}(A_m^*-A^*)(AA^*+\alpha I)^{-1}y\\ &\quad+(A_m^*A_m+\alpha I)^{-1}A_m^*(AA^*-A_mA^*)(AA^*+\alpha I)^{-1}y\\ &= \alpha(A_m^*A_m+\alpha I)^{-1}(A_m^*-A^*)(AA^*+\alpha I)^{-1}A\hat{x}\\ &\quad+R_{\alpha,m}(AA^*-A_mA^*)(AA^*+\alpha I)^{-1}y\\ &= \alpha(A_m^*A_m+\alpha I)^{-1}(A_m^*-A^*)A(A^*A+\alpha I)^{-1}\hat{x}\\ &\quad+R_{\alpha,m}(A-A_m)A^*(AA^*+\alpha I)^{-1}A\hat{x}.\end{aligned}$$

Using the property $A_m^*(I-P_m)=0$, we have

$$\begin{aligned}R_{\alpha,m}(A-A_m)A^* &= (A_m^*A_m+\alpha I)^{-1}A_m^*(A-A_m)A^*\\ &= (A_m^*A_m+\alpha I)^{-1}A_m^*(A-P_mA+P_mA-A_m)A^*\\ &= (A_m^*A_m+\alpha I)^{-1}A_m^*(P_mA-A_m)A^*.\end{aligned}$$

Hence, by using Lemmas 2.4 and 2.5 and the relations $\|(AA^*+\alpha I)^{-1}\|\leqslant\frac{1}{\alpha}$, $\|(AA^*+\alpha I)^{-1}A\|\leqslant\frac{1}{2\sqrt{\alpha}}$, with $A=A_m$, we have

$$\begin{aligned}\big\|(R_{\alpha,m}-R_\alpha)y\big\| &\leqslant \big\|(A_m^*-A^*)A\big\|\big\|(A^*A+\alpha I)^{-1}\hat{x}\big\|\\ &\quad+\frac{1}{2\sqrt{\alpha}}\big\|(P_mA-A_m)A^*\big\|\big\|(AA^*+\alpha I)^{-1}A\hat{x}\big\|.\end{aligned}\tag{2.13}$$

Now by the assumption $\hat{x}\in R((A^*A)^\nu)$, $0<\nu\leqslant 1$, and the spectral theory results

$$\big\|(A^*A+\alpha I)^{-1}\hat{x}\big\|\leqslant\|\hat{u}\|\alpha^{\nu-1}\quad\text{and}\quad\big\|(AA^*+\alpha I)^{-1}A\hat{x}\big\|\leqslant\|\hat{v}\|\alpha^{\omega-1},$$

where $\hat{u}$ and $\hat{v}$ are such that

$$\hat{x}=(A^*A)^\nu\hat{u}\quad\text{and}\quad\hat{x}=(A^*A)^{\omega-1}\hat{v},\quad 0<\nu\leqslant 1,\ \text{with } \omega=\min\{\nu+1/2,\ 1\}.$$

We see that if A satisfies the smoothness property S_1, then

$$\big\|(R_{\alpha,m}-R_\alpha)y\big\|\leqslant c\big(m2^{-mr}\alpha^{\nu-1}+m2^{-mr}\alpha^{\omega-3/2}\big)\tag{2.14}$$

and if A satisfies the smoothness property S_2, then

$$\big\|(R_{\alpha,m}-R_\alpha)y\big\|\leqslant c\big(2^{-mr}\alpha^{\nu-1}+2^{-mr}\alpha^{\omega-3/2}\big).\qquad\square\tag{2.15}$$

We now state the main theorem, giving an estimate for $\|\hat{x}-\tilde{x}_{\alpha,n}\|$ which is crucial for the result in the next section as well.

Theorem 2.7. *Suppose $y\in R(A)$ and A satisfies the smoothness property S_1. If $\hat{x}\in R((A^*A)^\nu)$, $0<\nu\leqslant 1$, then*

$$\|\hat{x}-\tilde{x}_{\alpha,m}\|\leqslant c\left(\alpha^\nu+\frac{\delta}{\sqrt{\alpha}}+\frac{m2^{-mr}}{\alpha}\alpha^\nu+\frac{m2^{-mr}}{\alpha}\alpha^{\omega-1/2}\right),$$

where $\omega=\min\{\nu+1/2,\ 1\}$.

In particular, if $0<\nu\leqslant 1/2$, then

$$\|\hat{x}-\tilde{x}_{\alpha,m}\|\leqslant c\left(\alpha^\nu+\frac{\delta}{\sqrt{\alpha}}+\frac{m2^{-mr}}{\alpha}\alpha^\nu\right).$$

If $1/2 \leqslant \nu \leqslant 1$, *then*

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leqslant c\left(\alpha^{\nu} + \frac{\delta}{\sqrt{\alpha}} + \frac{m2^{-mr}}{\sqrt{\alpha}}\right).$$

Proof. Recall from (2.12) that

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leqslant \|\hat{u}\|\alpha^{\nu} + \frac{\delta}{2\sqrt{\alpha}} + \left\|(R_\alpha - R_{\alpha,m})y\right\|.$$

Hence by using the estimate for $\|(R_\alpha - R_{\alpha,m})y\|$ given in (2.14), we get the required result. □

From the above theorem, we derive the following result with an a priori choice of the regularization parameter.

Theorem 2.8. *Let the assumptions in Theorem* 2.7 *be satisfied. For* $0 < \nu \leqslant 1/2$, *if* $\alpha := \alpha(\delta)$ *and* $m := m(\delta)$ *are such that*

$$\alpha \sim \delta^{\frac{2}{2\nu+1}}, \qquad m2^{-mr} \sim \delta^{\frac{2}{2\nu+1}},$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = O\left(\delta^{\frac{2\nu}{2\nu+1}}\right).$$

For $1/2 \leqslant \nu \leqslant 1$, *if* $\alpha := \alpha(\delta)$ *and* $m := m(\delta)$ *are such that*

$$\alpha \sim \delta^{\frac{2}{2\nu+1}}, \qquad m2^{-mr} \sim \delta,$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = O\left(\delta^{\frac{2\nu}{2\nu+1}}\right).$$

Proof. Proof follows from the estimate given in Theorem 2.7. □

If one uses the second set of smooth properties S_2, one can get a better result as given below.

Theorem 2.9. *Suppose* $y \in R(A)$, A *satisfies the smoothness property* S_2 *and if* $\hat{x} \in R((A^*A)^{\nu})$, $0 < \nu \leqslant 1$, *then*

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leqslant c\left(\alpha^{\nu} + \frac{\delta}{\sqrt{\alpha}} + \frac{2^{-mr}}{\alpha}\alpha^{\nu} + \frac{2^{-mr}}{\alpha}\alpha^{\omega-1/2}\right),$$

where $\omega = \min\{\nu + 1/2, 1\}$.

In particular, if $0 < \nu \leqslant 1/2$, *then*

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leqslant c\left(\alpha^{\nu} + \frac{\delta}{\sqrt{\alpha}} + \frac{2^{-mr}}{\alpha}\alpha^{\nu}\right)$$

and if $\alpha := \alpha(\delta)$ *and* $m := m(\delta)$ *are such that*

$$\alpha \sim \delta^{\frac{2}{2\nu+1}}, \qquad 2^{-mr} \sim \delta^{\frac{2}{2\nu+1}},$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = O\big(\delta^{\frac{2\nu}{2\nu+1}}\big).$$

If $1/2 \leqslant \nu \leqslant 1$, *then*

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| \leqslant c\left(\alpha^{\nu} + \frac{\delta}{\sqrt{\alpha}} + \frac{2^{-mr}}{\sqrt{\alpha}}\right)$$

and if $\alpha := \alpha(\delta)$ *and* $m := m(\delta)$ *are such that*

$$\alpha \sim \delta^{\frac{2}{2\nu+1}}, \qquad 2^{-mr} \sim \delta,$$

then

$$\|\hat{x} - \tilde{x}_{\alpha,m}\| = O\big(\delta^{\frac{2\nu}{2\nu+1}}\big).$$

3. Discussion on computational complexity

In this section, we discuss the advantage of the method (1.9) over the known projection methods in terms of their computational complexities. The system to be solved in each case has the form

$$(A_N^* A_N + \alpha I)\tilde{x}_{\alpha,N} = A_N^* \tilde{y}, \quad \alpha > 0, \tag{3.1}$$

where A_N is a finite rank approximation of A, obtained using projections. We compare the computational complexity in terms of number of inner products required to solve the above system with different A_N's. For convenience, we use the following notation:

Card I.P: number of inner products.

Case I. $A_N = P_m A P_m$. In this case, we will be solving an $m \times m$ system and the number of inner products used for computation is

$$Card\, I.P = \dim(V_m) \cdot \dim(V_m) \sim 2^{2ms}.$$

If $\hat{x} \in R((T^*T)^{\nu})$, then from Theorem 3.1 of Plato and Vainikko [6] and Lemma 2.1, it follows that

$$2^m \sim \delta^{\frac{-2\nu}{r(2\nu+1)p}}, \quad \text{where } p = \min\{2\nu, 1\}.$$

If $0 < \nu \leqslant 1/2$, then $p = 2\nu$ so that $Card\, I.P \sim \delta^{\frac{-2s}{(2\nu+1)r}}$. When $1/2 < \nu \leqslant 1$, $p = 1$ so that $Card\, I.P \sim \delta^{\frac{-4s\nu}{(2\nu+1)r}}$. Therefore, solving (3.1) with $P_m A P_m$

$$Card\, I.P \sim \begin{cases} \delta^{\frac{-2s}{(2\nu+1)r}} & \text{if } 0 < \nu \leqslant 1/2, \\ \delta^{\frac{-4s\nu}{(2\nu+1)r}} & \text{if } 1/2 < \nu \leqslant 1. \end{cases}$$

Case II. $A_N = A_m = A P_m$. Our interest is to seek a solution of the form $\tilde{x}_{\alpha,m} = \sum_{i=1}^{\dim(V_m)} x_i e_i$ by solving (3.1) with $A_N = A P_m$. This is equivalent to solving a system

$$\alpha x_i + \sum_{j=1}^{\dim(V_m)} \langle A e_j, A e_i \rangle x_j = \langle \tilde{y}, A e_i \rangle, \quad i = 1, \ldots, \dim(V_m).$$

Here inner products of the form $\langle Ae_j, Ae_i\rangle$ and $\langle\tilde{y}, Ae_i\rangle$ are used and the total number of inner products required are

$$Card\ I.P = \dim(V_m) + \dim(V_m)\cdot\dim(V_m) \sim \dim(V_m)\cdot\dim(V_m) \sim 2^{2sm}.$$

If $\hat{x} \in R((A^*A)^\nu)$, then using the relation (2.10) of George and Nair [2] and Lemma 4.3 of Plato and Vainikko [6], it is seen that for getting the optimal order $O(\delta^{2\nu/(2\nu+1)})$, the condition required on m with A satisfying the smoothness property (1.7) is that $2^m \sim \delta^{\frac{-2\nu}{r(2\nu+1)p}}$, where $p = \min\{2\nu,\ 1\}$.

If $0 < \nu \leqslant 1/2$, then $p = 2\nu$ and when $1/2 < \nu \leqslant 1$, $p = 1$. Hence,

$$Card\ I.P \sim \begin{cases} \delta^{\frac{-2s}{(2\nu+1)r}} & \text{if } 0 < \nu \leqslant 1/2, \\ \delta^{\frac{-4s\nu}{(2\nu+1)r}} & \text{if } 1/2 < \nu \leqslant 1. \end{cases}$$

Case III. $A_N = A_m = P_0AP_m + \sum_{k=1}^m (P_k - P_{k-1})AP_{m-k}$. In this case, the inner products involved are $\langle e_i, \tilde{y}\rangle$ and $\langle e_i, Ae_j\rangle$ for $(i, j) \in \Omega_m = \{0\} \times [0, m] \bigcup_{k=1}^m (k-1, k] \times [0, m-k]$.

$$Card\ I.P \sim \sum_{k=0}^{m} \dim(V_k)\cdot\dim(V_{m-k}) + \dim(V_m) \sim \sum_{k=0}^{m} 2^{ks}\cdot 2^{(m-k)s} + 2^{ms} \sim m2^{ms}.$$

If $\hat{x} \in R((A^*A)^\nu)$, then by Theorem 2.8, the condition on m for obtaining the optimal order $O(\delta^{\frac{2}{2\nu+1}})$ is that $m2^{-mr} \sim \delta^{\frac{2}{2\nu+1}}$, for $0 < \nu \leqslant 1/2$ and $m2^{-mr} \sim \delta$, for $1/2 \leqslant \nu \leqslant 1$. Therefore, for solving (3.1) with A_m, the requirements are

$$Card\ I.P \sim \begin{cases} \delta^{\frac{-2s}{(2\nu+1)r}} (\log(\delta^{-1}))^{1+s/r} & \text{if } 0 < \nu \leqslant 1/2, \\ \delta^{\frac{-s}{r}} (\log(\delta^{-1}))^{1+s/r} & \text{if } 1/2 < \nu \leqslant 1. \end{cases}$$

4. Numerical experiments

Though our main purpose was to study the convergence analysis of the discretization scheme, we carry out some numerical experiments in this section to illustrate that the proposed scheme is implementable.

Consider the Hilbert space $X = L^2[0, 1]$ and $\{e_1, e_2, \ldots\}$, the Haar orthonormal basis of piecewise constant functions, where $e_1(t) = 1$ for all $t \in [0, 1]$, and for $m = 2^{k-1} + j$, $k = 1, 2, \ldots,\ j = 1, 2, \ldots, 2^{k-1}$,

$$e_m(t) = \begin{cases} 2^{\frac{k-1}{2}} & \text{if } t \in \left[\frac{j-1}{2^{k-1}}, \frac{j-1/2}{2^{k-1}}\right), \\ -2^{\frac{k-1}{2}} & \text{if } t \in \left[\frac{j-1/2}{2^{k-1}}, \frac{j}{2^{k-1}}\right), \\ 0 & \text{if } t \notin \left[\frac{j-1}{2^{k-1}}, \frac{j}{2^{k-1}}\right]. \end{cases}$$

Let $A : X \to X$ be the integral operator,

$$(Ax)(s) = \int_0^1 k(s, t)x(t)\,dt, \quad s \in [0, 1],$$

with the kernel

$$k(s,t) = \begin{cases} t(1-s) & \text{if } t \leqslant s, \\ s(1-t) & \text{if } t > s. \end{cases}$$

We take X^r with $r = 1$ as the Sobolev space of functions f with derivative $f' \in L^2[0,1]$. We consider two examples for our discussion.

Example 1. Let $y(s) = \frac{1}{24}(s - 2s^3 + s^4)$. In this case, it can be seen that $\hat{x}(t) = \frac{1}{2}(t - t^2)$, $t \in [0,1]$. It is known (cf. [3]) that $\hat{x} \in R(T^*T)^\nu$ for all $\nu < \frac{5}{8}$.

Example 2. Let $y(s) = \frac{1}{30}(3s - 5s^3 + 3s^5 - s^6)$. In this case, it can be seen that $\hat{x}(t) = t - 2t^3 + t^4$, $t \in [0,1]$. It is known (cf. [3]) that $\hat{x} \in R(T^*T)^\nu$ for all $\nu < \frac{9}{8}$.

In the first example, we take $\nu = 1/2$ and hence by Theorem 2.8, choosing $\alpha \sim \delta$ and $m2^{-m} \sim \delta$, we get the optimal rate $\|\hat{x} - \tilde{x}_{\alpha,m}\| = O(\delta^{1/2})$.

In the second example, we take $\nu = 1$ and hence by Theorem 2.8, choosing $\alpha \sim \delta^{2/3}$ and $m2^{-m} \sim \delta$, we get the optimal order $\|\hat{x} - \tilde{x}_{\alpha,m}\| = O(\delta^{2/3})$.

For illustration, we take $s = 1$ and a randomly perturbed data $\tilde{y}$ with $\|\tilde{y} - y\| \leqslant \delta$. We consider two cases with data error as 10% and 15%. For computing the solution, we followed LU decomposition techniques. Probably, one could use an iterative method for implementation. The numerically computed $\|\hat{x} - \tilde{x}_{\alpha,m}\|$ is denoted by $\tilde{e}_{\alpha,m}$. Actual data for Examples 1 and 2 are shown in Figs. 1 and 2, respectively. Perturbed data with an error of 10% and 15% for Example 1 is shown in Figs. 3 and 4 and, for Example 2 is given in Figs. 5 and 6, respectively. Numerical results of Example 1 are given in Table 1 and that of Example 2 is given in Table 2. The number of inner products used for solving a 64×64 system in the case of new scheme is 256 whereas that of the other two cases is 4096. Similarly, for solving a 128×128 system, the new scheme used 576 inner products and in other cases, it is 16384. In the case of new scheme, computed solution plotted against the actual solution when data error is 10% is shown in Fig. 7 for Example 1 and that for Example 2 is given in Fig. 8. A combined solution obtained in different cases for Examples 1 and 2 are respectively given in Figs. 9 and 10. Such solutions are obtained by solving a system of size 64×64. Similarly when data error is 15%, the new scheme solution for Examples 1 and 2 are given in Figs. 11 and 12. The respective combined solutions are shown in Figs. 13 and 14. In the this case, solutions are obtained by solving a system of size 128×128. During the numerical experiments, it is observed that choice of α is very significant for getting a more accurate solution. We have chosen α depends on δ namely, $\alpha \sim \delta$ in Example 1 and $\alpha \sim \delta^{2/3}$ in Example 2. One pertinent question is that in practice, if one does not have the actual knowledge about the data error, how one would choose the parameter α? In such cases, it is desirable to choose α that depends on m. This fact is more clear from Theorem 2.8 where we can choose α depends on m too. Had α been chosen depending on m in our examples namely, $\alpha \sim m2^{-m}$ in Example 1 and $\alpha \sim (m2^m)^{2/3}$ in Example 2 we would have obtained similar results. This is illustrated in Table 3. From the numerical results, it is evident that our scheme achieves the same result with less discrete information compared with other traditional schemes.

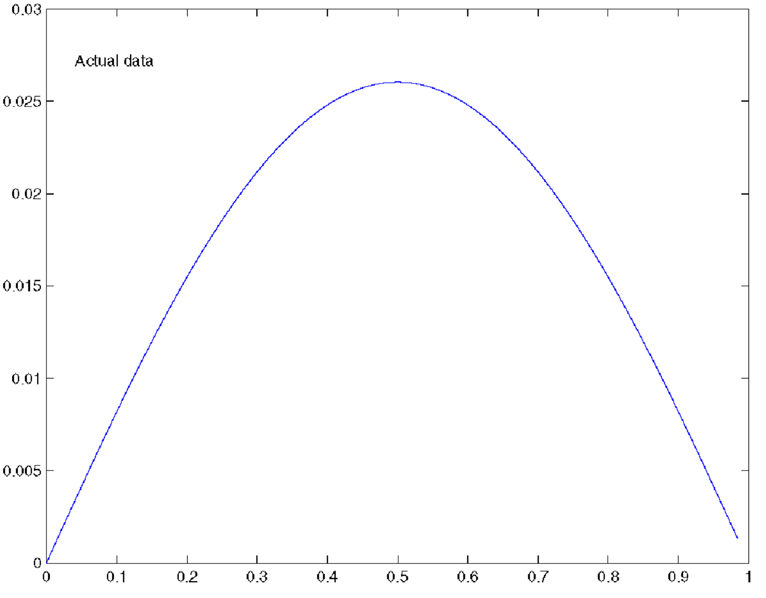


Fig. 1. Actual data (Example 1).

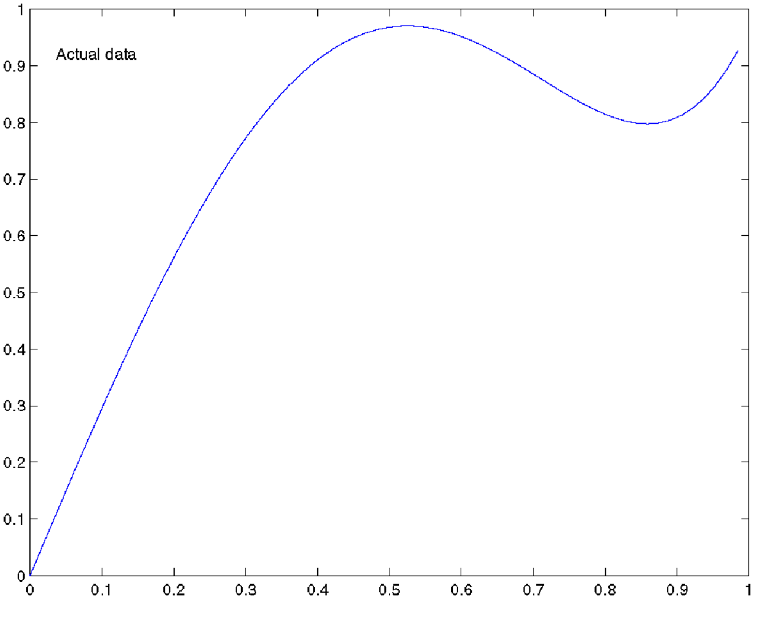


Fig. 2. Actual data (Example 2).

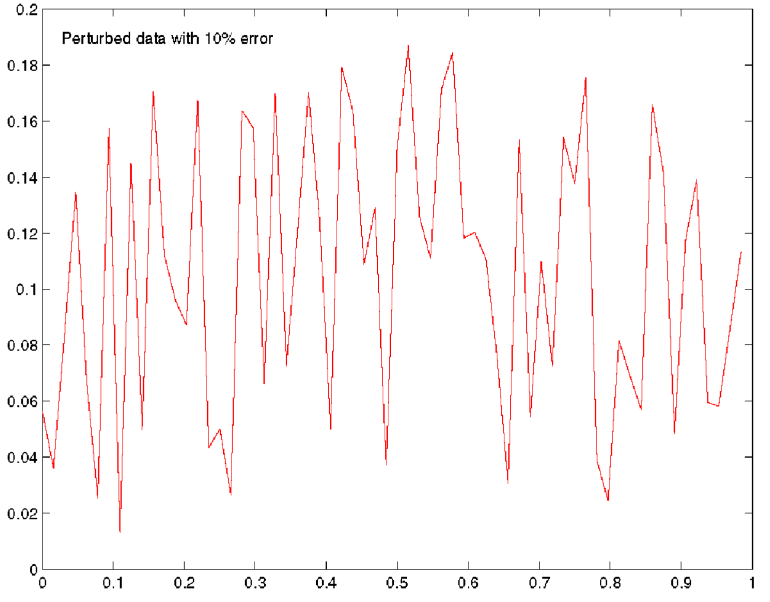


Fig. 3. 10% perturbed data (Example 1).

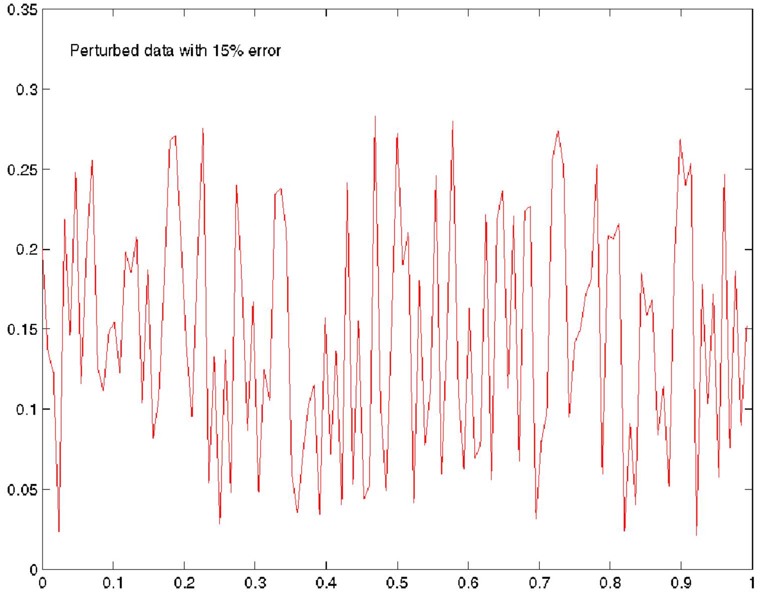


Fig. 4. 15% perturbed data (Example 1).

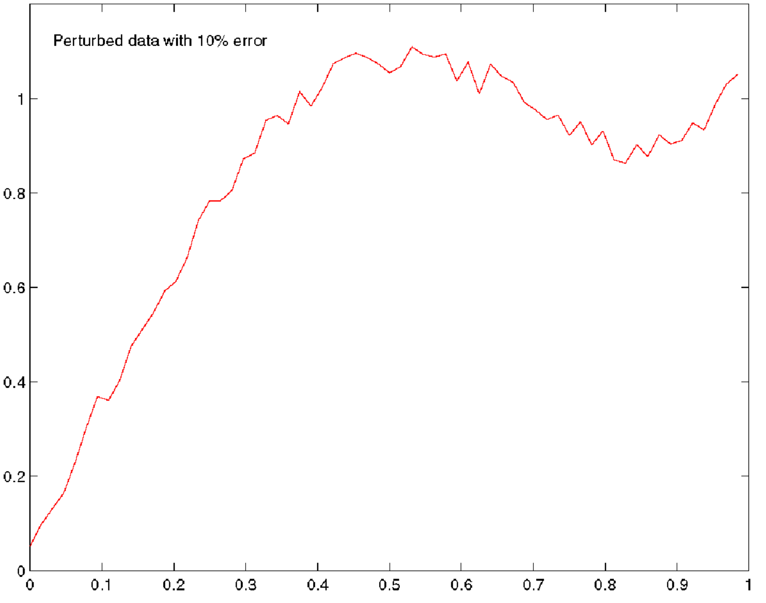


Fig. 5. 10% perturbed data (Example 2).

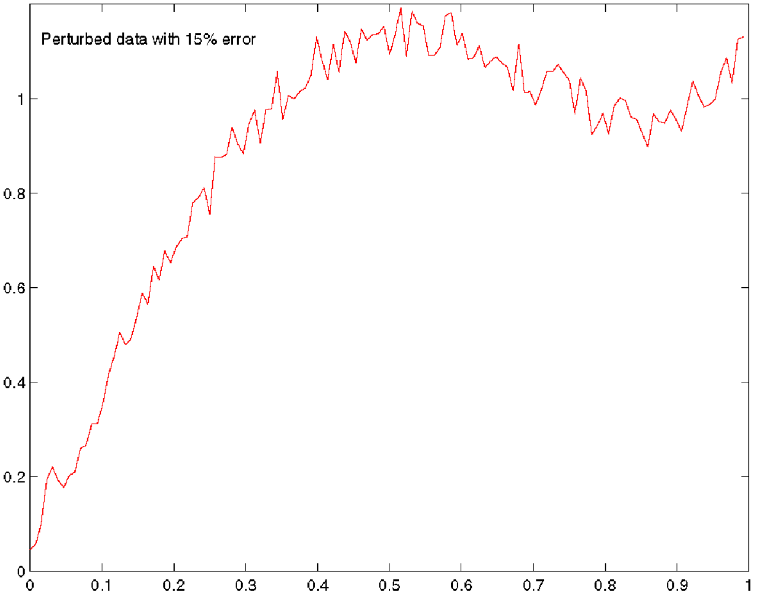


Fig. 6. 15% perturbed data (Example 2).

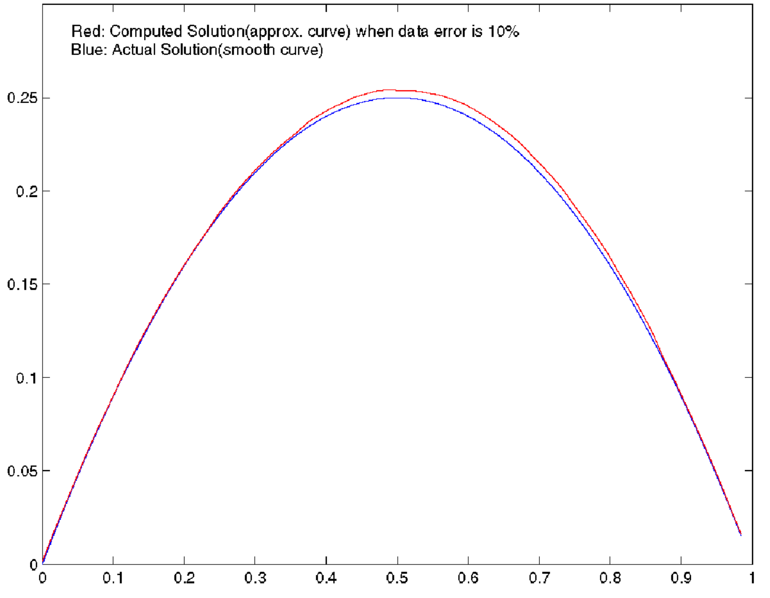


Fig. 7. New scheme solution, $\delta = 10\%$ (Example 1).

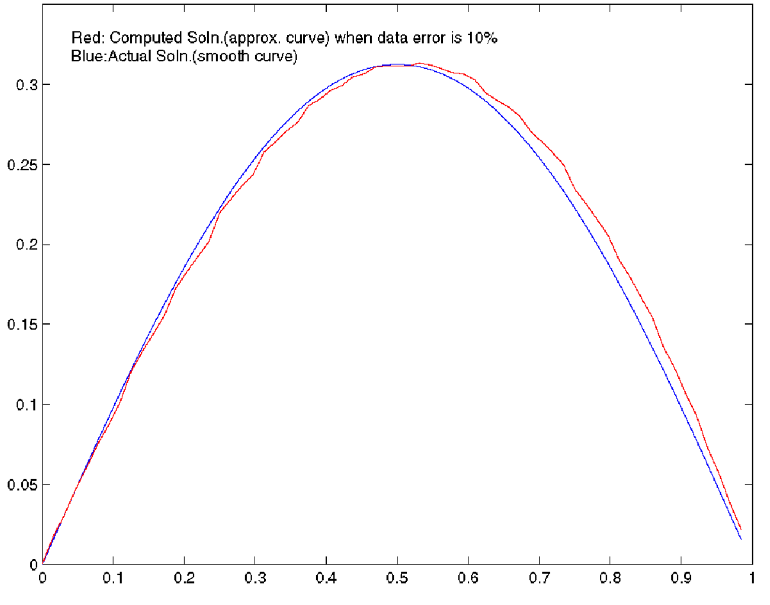


Fig. 8. New scheme solution, $\delta = 10\%$ (Example 2).

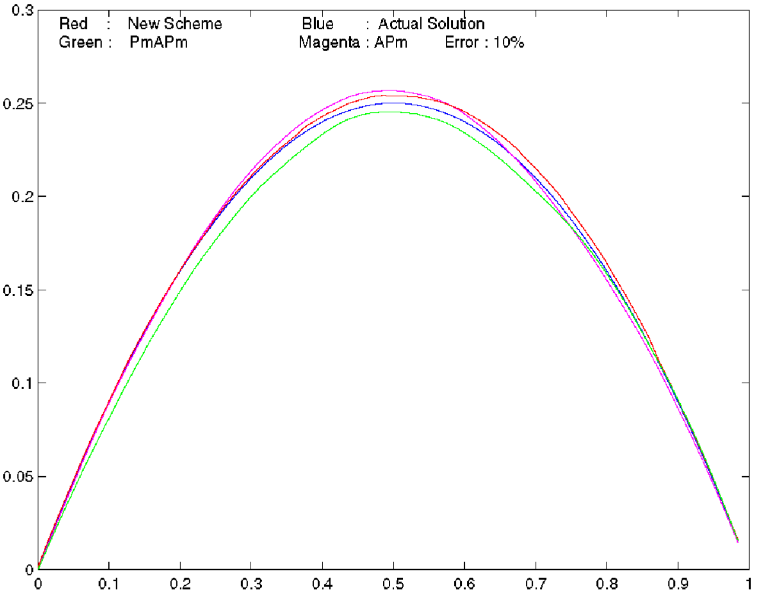


Fig. 9. Combined solution, $\delta = 10\%$ (Example 1).

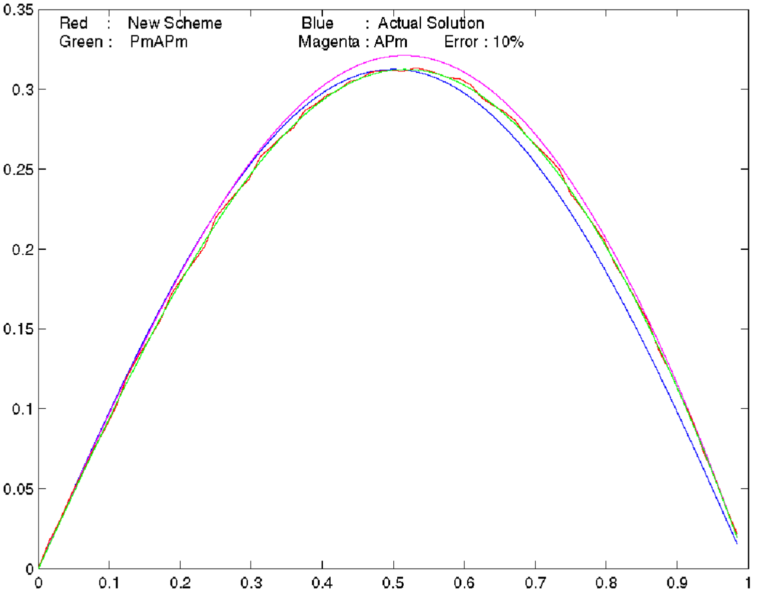


Fig. 10. Combined solution, $\delta = 10\%$ (Example 2).

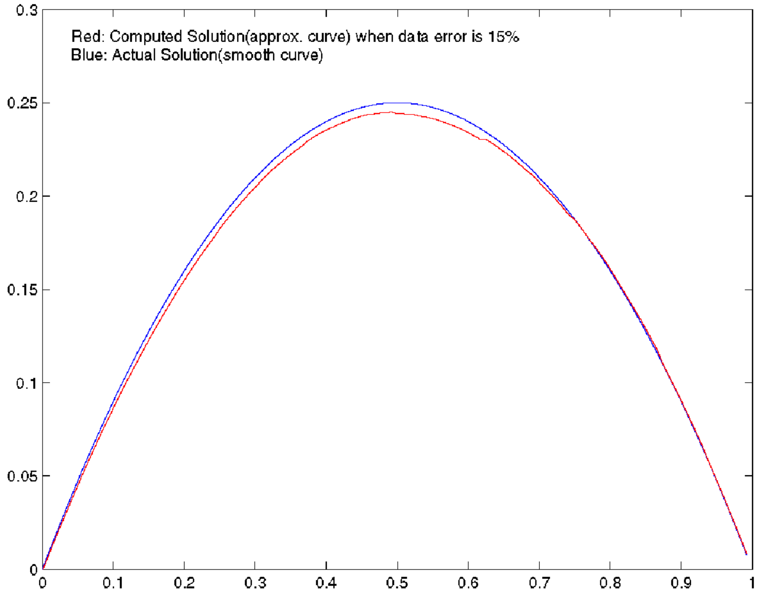


Fig. 11. New scheme solution, $\delta = 15\%$ (Example 1).

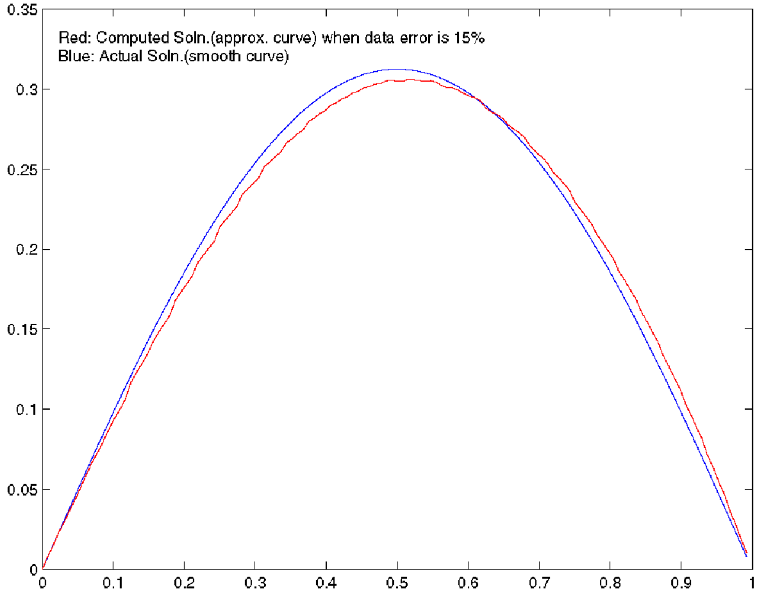


Fig. 12. New scheme solution, $\delta = 15\%$ (Example 2).

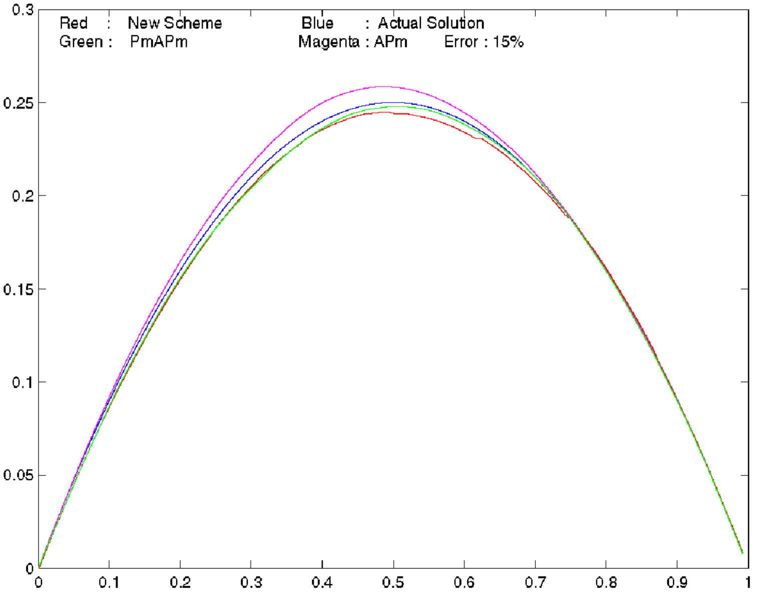


Fig. 13. Combined solution, $\delta = 15\%$ (Example 1).

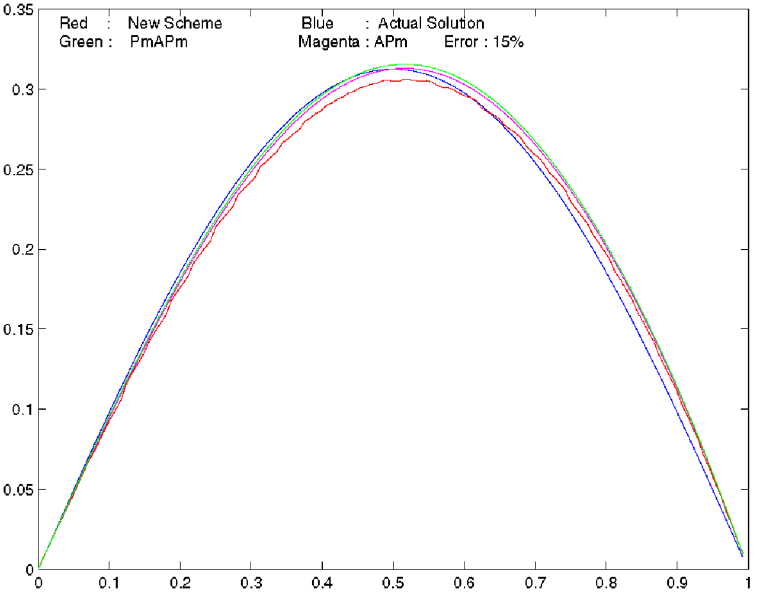


Fig. 14. Combined solution, $\delta = 15\%$ (Example 2).

Table 1
An a priori parameter choice results for Example 1, when $\nu = 0.5$

δ (%)	α	Method	Dimension	$\tilde{e}_{\alpha,m}$	$\tilde{e}_{\alpha,m} \cdot \delta^{-2\nu/(2\nu+1)}$
10	0.045	New scheme	4	0.0363929643	0.1150846580
			16	0.0135003285	0.0426917873
			64	0.0086975775	0.0275041550
	0.045	$P_m A P_m$	4	0.1266658437	0.4005525680
			16	0.0924377012	0.2923136775
			64	0.0857556078	0.2711883042
	0.055	$A P_m$	4	0.0562288426	0.1778112129
			16	0.0038163523	0.0120683658
			64	0.0019646451	0.0062127533
15	0.0675	New scheme	8	0.0184732466	0.0476977176
			32	0.0155431933	0.0401323525
			128	0.0017517220	0.0045229269
	0.0675	$P_m A P_m$	8	0.1190329991	0.3073418822
			32	0.0975507585	0.251874977
			128	0.0904135271	0.2334467234
	0.075	$A P_m$	8	0.0179760834	0.0464140480
			32	0.0096494985	0.0249148981
			128	0.0061330988	0.0158355931

Table 2
An a priori parameter choice results for Example 2, when $\nu = 1$

δ (%)	α	Method	Dimension	$\tilde{e}_{\alpha,m}$	$\tilde{e}_{\alpha,m} \cdot \delta^{-2\nu/(2\nu+1)}$
10	0.367729	New scheme	4	0.0182970682	0.0855818087
			16	0.0120169553	0.0562079239
			64	0.0088719118	0.0414970470
	0.367729	$P_m A P_m$	4	0.0105979717	0.0495704379
			16	0.0090166016	0.0421738143
			64	0.0084846678	0.0396857730
	0.367729	$A P_m$	4	0.0176648221	0.0826245807
			16	0.0119232599	0.0557692766
			64	0.0113697662	0.0531803922
15	0.398354	New scheme	8	0.0190867559	0.0680379119
			32	0.0103900837	0.0370371494
			128	0.0090276059	0.0321804009
	0.384327	$P_m A P_m$	8	0.0112103314	0.0399610885
			32	0.0099913244	0.0356157060
			128	0.0094762502	0.0337796679
	0.398354	$A P_m$	8	0.0098039750	0.0349478955
			32	0.0091289653	0.0325414996
			128	0.0083796992	0.0298708297

Table 3
An a priori parameter choice results for Examples 1 and 2

δ (%)	Example	α	Dimension	$\tilde{e}_{\alpha,m}$	$\tilde{e}_{\alpha,m} \cdot \delta^{-2\nu/(2\nu+1)}$
10	Example 1	0.235	4	0.1374539512	0.4346675593
		0.1175	16	0.1105106951	0.3494655002
		0.0440625	64	0.0029936074	0.0094666179
10	Example 2	1.106171	4	0.1463864827	0.6847010220
		0.695236	16	0.1030748477	0.4821172834
		0.360357	64	0.0108394062	0.0506497120

Conclusion

In this paper, we carried out the convergence analysis of a finite-dimensional approximation scheme and compared it with traditional projection schemes. From our theoretical as well as numerical investigation, it is evident that this scheme has an edge over other traditional schemes in terms of the amount of discrete information for solving the problem. Though we carried out the complexity analysis only with the smoothness property S_1, similar conclusions can also be drawn if one uses the smoothness property S_2. Extending this work to a class of regularization methods and a posteriori parameter choice strategy is under investigation.

Acknowledgments

I thank the referees for their useful comments which significantly improved the presentation of the paper.

References

[1] V. Dicken, P. Maass, Wavelet–Galerkin methods for ill-posed problems applications, J. Inverse Ill-Posed Probl. 87 (1996) 203–221.
[2] S. George, M.T. Nair, On a generalized Arcangeli's method for Tikhonov regularization with inexact data, Numer. Funct. Anal. Optim. 19 (1998) 773–787.
[3] H. Gfrerer, An a posteriori parameter choice for ordinary and iterated Tikhonov regularization of ill-posed problems leading to optimal convergence rates, Math. Comput. 49 (1987) 507–522, S5–S12.
[4] C.W. Groetsch, The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind, Pitman, London, 1984.
[5] P. Maass, S. Pereverzev, R. Ramlau, S.G. Solodkdy, An adaptive discretization for Tikhonov–Phillips regularization with a posteriori parameter selection, Numer. Math. 87 (2001) 485–502.
[6] R. Plato, G. Vainikko, On the regularization of projection methods for solving ill-posed problems, Numer. Math. 57 (1990) 63–79.
[7] E. Schock, Integral equations of third kind, Studia Math. 81 (1985) 1–11.